S5.1 Solution

S 5.1 (a) $\frac{d^2y}{dt^2} + y = 0$

(i) let $x_1 = y$, $x_2 = \frac{dy}{dt}$

then $\dot{x}_1 = \frac{dy}{dt} = x_2$

$\dot{x}_2 = \frac{d^2y}{dt^2} = -y = -x_1$

$$\boxed{\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}$$

$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

(ii) $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ Eigenvalues satisfy $\det[\lambda I - A] = 0$

$\begin{vmatrix} \lambda & -1 \\ 1 & \lambda \end{vmatrix} = \lambda^2 + 1 = 0$

$\Rightarrow \boxed{\lambda = \pm j}$

Eigenvectors: $\lambda = j \Rightarrow \begin{bmatrix} j & -1 \\ 1 & j \end{bmatrix} \vec{v}_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \boxed{\vec{v}_1 = \begin{bmatrix} 1 \\ j \end{bmatrix}}$

$\boxed{\vec{v}_2 = \begin{bmatrix} 1 \\ -j \end{bmatrix}}$

(Real matrix $\Rightarrow$ complex conjugate eigenvalues & eigenvectors)

(iii) Second-order system $\Rightarrow$ need $\boxed{\text{2 initial conditions}}$

e.g. $\boxed{\text{on } y(0) \text{ and } \frac{dy}{dt}(0)}$

(iv) General form of $\vec{x}(t)$ is $\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ j \end{bmatrix} e^{jt} + c_2 \begin{bmatrix} 1 \\ -j \end{bmatrix} e^{-jt}$

$\Rightarrow$ general form of $y(t)$ is $\boxed{y(t) = A\cos t + B\sin t}$

where A, B are constants determined by the initial conditions

(could also see this from the original d.e.)

S5.1 Solution

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S5.1 (b) (i) Node P : $i_1 - i_2 + i_3 = 0$

Node Q : $-i_1 + i_2 - i_3 = 0$

(ii) Right loop : $10 i_2 + 10 i_3 + 15 i_3 = 90$

$10 i_2 + 25 i_3 = 90$

Left loop : $20 i_1 + 10 i_2 = 80$

(iii) 3 unknowns, 4 equations $\Rightarrow$ A is 4×3

$$\underset{(4\times3)}{\begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 0 & 10 & 25 \\ 20 & 10 & 0 \end{bmatrix}} \underset{(3\times1)}{\begin{bmatrix} i_1 \\ i_2 \\ i_3 \end{bmatrix}} = \underset{(4\times1)}{\begin{bmatrix} 0 \\ 0 \\ 90 \\ 80 \end{bmatrix}}$$

(iv) Augmented matrix is:

$$\begin{array}{c} r_1 \\ r_2 \\ r_3 \\ r_4 \end{array} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ -1 & 1 & -1 & 0 \\ 0 & 10 & 25 & 90 \\ 20 & 10 & 0 & 80 \end{array}\right]$$

$$\rightarrow \begin{array}{c} r_1 \\ r_2 + r_1 \\ r_3 \\ r_4 - 20r_1 \end{array} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 10 & 25 & 90 \\ 0 & 30 & -20 & 80 \end{array}\right] \xrightarrow[\text{to bottom}]{\text{switch row 2}} \begin{array}{c} r_1' \\ r_2' \\ r_3' \\ r_4' \end{array} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 10 & 25 & 90 \\ 0 & 30 & -20 & 80 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\longrightarrow \begin{array}{c} r_1' \\ r_2' \\ r_3' - 3r_2' \\ r_4' \end{array} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 10 & 25 & 90 \\ 0 & 0 & -95 & -190 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

This is in upper triangular form

Now solve with back substitution:

$-95 i_3 = -190 \Rightarrow i_3 = 2$

$10 i_2 = 90 - 25 i_3 = 40 \Rightarrow i_2 = 4$

$i_1 = 0 + i_2 - i_3 = 2 \Rightarrow i_1 = 2$

Solution : $\boxed{i_1 = 2A, \quad i_2 = 4A, \quad i_3 = 2A}$